because $\mathbf{R}$ is also a constant vector with respect to the coordinate system $\Sigma_1$ fixed on the body. Combining (7) and (8) we obtain

$$\boldsymbol{\omega}_1 \times \mathbf{R} = \boldsymbol{\omega}_1 \times \mathbf{r}_1 - \boldsymbol{\omega}_2 \times \mathbf{r}_2 \tag{9}$$

or, inasmuch as $\mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2$,

$$(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1) \times \mathbf{r}_2 = 0. \tag{10}$$

Since $P$ is any point of the rigid body, $\mathbf{r}_2$ is an arbitrary vector that can be freely changed without affecting either $\boldsymbol{\omega}_1$ or $\boldsymbol{\omega}_2$. As an immediate consequence $\boldsymbol{\omega}_2 = \boldsymbol{\omega}_1$, and the proof is complete.

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# Fourier transform solution to the semi-infinite resistance ladder

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In a recent article[1] a general method for solving for the resistance between any two nodes of a number of infinite resistance lattices using discrete-variable Fourier transforms[2] was presented. In this note the same technique is applied to the *semi-infinite* ladder network. The mathematical methods used in the solution to this problem (Fourier transforms and contour integration) are well within the abilities of the undergraduate physics major. This one-dimensional example is thus complementary to the two (and higher)-dimensional networks presented in the previous article,[1] and this example can be used in a junior-level mathematical methods course.

The input resistance of the semi-infinite ladder network composed of identical resistors, shown in Fig. 1, is well known and can be solved by using simple rules of parallel and series combinations of resistors (see the paper by Srinivasan[3] and references therein). One simply notes that adding on another resistive repeat unit to the semi-infinite ladder will not affect the overall input resistance so that $R_{\mathrm{eq}} = R'_{\mathrm{eq}}$. The equivalent resistance is thus found to be equal to the golden ratio multiplied by the unit of resistance, $R_{\mathrm{eq}} = [(1+\sqrt{5})/2]R$.

This result can be found using the Fourier transform solution to the difference equation governing the auxiliary resistance ladder shown in Fig. 2(a). This auxiliary ladder, which we introduce for mathematical convenience, is infinite in both directions, whereas the ladder of primary interest is semi-infinite (infinite in only one direction). We can easily relate the currents and voltages of the infinite ladder to those of the semi-infinite ladder. If we know that a node voltage, $v_0$, results from the input of 1 A of current at the same node, then the resistance, $R_{\mathrm{eq}}$, can be found, as shown in Fig. 2(b). Simple application of the current rule to Fig. 2(b) yields

$$1 = \frac{v_0}{R} + \frac{2v_0}{R_{\mathrm{eq}}} \Rightarrow R_{\mathrm{eq}} = \frac{2v_0 R}{R - v_0}. \tag{1}$$

We can determine the node voltage at any node, $n$, based on the current entering that node in the infinite ladder of Fig.

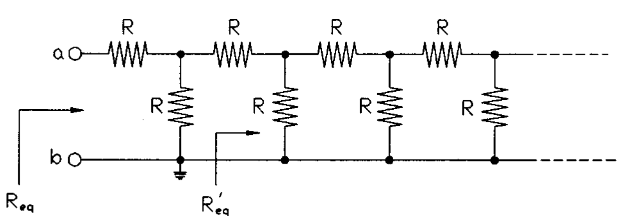


Fig. 1. The semi-infinite resistor ladder. The resistance between nodes a and b is defined as the input resistance for this network.

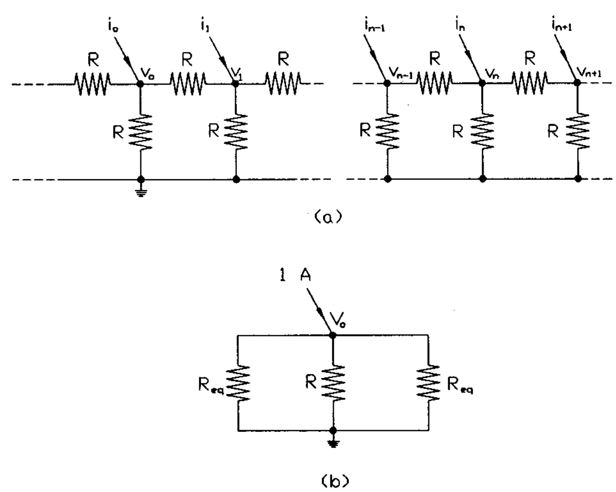


Fig. 2. Reduction of (a) the infinite resistor ladder to (b) a circuit in terms of the equivalent resistance between nodes a and b of the semi-infinite ladder shown in Fig. 1. Note that all of the external currents being fed into the network in (a) are set to zero except $i_0 = 1$ A for the specific case of making the equivalence to the circuit shown in Fig. 1.

  

2(a). Kirchhoff's current law applied to node $n$ results in the following difference equation,

$$Ri_n = 3v_n - v_{n+1} - v_{n-1}, \tag{2}$$

where $i_n$ is current fed into node $n$ supplied by an external source.

Following the method presented in Atkinson's paper,[1] we use the discrete-variable Fourier transform pair (also called a complex Fourier series),

$$x_n = \frac{1}{2\pi}\int_0^{2\pi} d\beta\, X(\beta) e^{in\beta} \leftrightarrow X(\beta) = \sum_{n=-\infty}^{\infty} x_n e^{-in\beta}, \tag{3}$$

and transform the difference equation from $n$ to $\beta$. The resulting transformed voltage is

$$v(\beta) = \frac{RI(\beta)}{3-2\cos\beta}, \tag{4}$$

with the actual voltage at node $n$ expressed as the integral

$$v_n = \frac{R}{2\pi}\int_0^{2\pi} d\beta \frac{I(\beta)}{3-2\cos\beta} e^{in\beta}. \tag{5}$$

A current of 1 A fed into node 0 as shown in Fig. 2, represented as $i_n = \delta_{n,0}$ (where $\delta_{n,k}=1$ if $n=k$ and 0 otherwise), results in a transformed current of $I(\beta)=1$. Therefore, the voltage at any node, $n$, under these circumstances is written as

$$v_n = \frac{R}{2\pi}\int_0^{2\pi} d\beta \frac{e^{in\beta}}{3-2\cos\beta}. \tag{6}$$

The voltage at node 0, $v_0$, can be evaluated readily using contour integration. The integral which we wish to solve is

$$v_0 = \frac{R}{2\pi}\int_0^{2\pi} \frac{d\beta}{3-2\cos\beta}, \tag{7}$$

with the corresponding contour integral

$$v_0 = \frac{R}{2\pi i}\oint_{|z|=1} \frac{dz}{z} \frac{1}{3-(z+z^{-1})}. \tag{8}$$

The integrand has only one pole located in the unit circle, $z=(3-\sqrt{5})/2$. Therefore, the integral evaluates to $v_0 = R/\sqrt{5}$, which is the effective resistance of the infinite ladder. Finally, we obtain the effective resistance of the semi-infinite ladder by substitution of $v_0$ into Eq. (1) and obtain $R_{\mathrm{eq}} = [(1+\sqrt{5})/2]R$.


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# Comment on ''Ideal capacitor circuits and energy conservation,'' by K. Mita and M. Boufaida [Am. J. Phys. 67 (8), 737–739 (1999)]


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K. Mita and M. Boufaida (hereinafter MB)[1] discuss the puzzle of the missing energy in a capacitor charged from a power supply (a battery or another capacitor), with neither resistance nor inductance in the circuit. In such a circuit, the power supply appears to deliver energy $E_{\mathrm{PS}} = qV_0$, while the capacitor only stores $\frac{1}{2}CV_0^2 = \frac{1}{2}qV_0$. The problem disappears if either inductance $L$ or resistance $R$ is introduced into the circuit, where $L$ and $R$ can be as small as one likes, but not zero (a rather peculiar discontinuity). MB note that the function of $L$ and/or $R$ is to change a discontinuous, instantaneous charging process into a continuous one, with a finite time constant. They then generalize their observation, and show that any power supply described by $V=V_0 f(t)$, where $f(t)$ is a monotonically increasing, continuous, differentiable function of time, will deliver the ''correct'' energy. With $0<t<t_0$, $f(0)=0$, $f(t_0)=1$, and $q=CV$,

$$E_{\mathrm{PS}} = \int \frac{dq}{dt} V\, dt = CV_0^2 \int f(t) \frac{df}{dt} dt = CV_0^2 \int f\, df = \tfrac{1}{2} CV_0^2. \tag{1}$$

MB note that the charge across the instantaneously charging capacitor can be written as $q = CV_0 \Theta(t)$, where $\Theta(t)$ is the Heaviside step-function: